Non-asymptotic Heisenberg scaling: experimental metrology for a wide resources range

The parameter to be estimated will be a phase (rotation angle).

**Mach-Zehnder Interferometer**: used to measure changes in the optical lengths of the arms.

What if we use truly quantum states of light? E.g. number states.
Mach-Zehnder interferometer

Single photon source

404nm laser pump

Periodically poled titanyl phosphate (ppKTP) crystal

Signal photon

Trigger photon

|10⟩ → \frac{|10⟩+|01⟩}{\sqrt{2}} → \frac{|10⟩+e^{i\theta}|01⟩}{\sqrt{2}}

Photodetectors

“Coin-toss variable”
Abstractly \[ |0\rangle + e^{i\theta} |1\rangle \] \[ \frac{1}{\sqrt{2}} \] with \( |0\rangle \) and \( |1\rangle \) being two orthogonal quantum states.

**Phase estimation is the prototypical quantum technological task!**

- Has its origin in the coherent nature of quantum mechanics, which allows for superposition with complex amplitudes.
- It’s prototypical for the nature of quantum information: \( \theta \in \mathbb{R} \) contains \( \infty \) bits of information, but a measurement can extract at most 1 bit. We have however many ways to extract such bit, corresponding to many measurements.

\[
|\psi_\theta\rangle = \frac{|0\rangle + e^{i\theta} |1\rangle}{\sqrt{2}}
\]
Roles of the Quantum Phase Estimation in...

Quantum Algorithms:

- Quantum Eigensolver
- HHL algorithm for linear systems
  (Harrow, Hassidim, and Lloyd, (2008)
  *Phys. Rev. Lett.* **103** (15): 150502)
- Shor factorization algorithm
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Quantum Metrology:

- Interferometry (gravitational waves)
  (Schnabel *et al.* Nat Commun **1**, 121 (2010))
- Concentration measurements in a solution
- Magnetometry
  (Budker and Romalis, Nature Physics **3**, 227–234 (2007))
- Characterization of quantum gates
- ...
Abstract point of view: single qubit encoded by a gate $\sigma_z$

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$$U_\theta := e^{i\sigma_z \theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

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We can define an estimator $\hat{\theta}$ from the results of multiple measurements, and define a **Quantum Cramér-Rao bound**.

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$$\Delta^2 \hat{\theta} = \mathbb{E}[(\hat{\theta} - \theta)^2] \geq \frac{1}{N}$$

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Heisenberg scaling:

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Maximally entangled states for Quantum Metrology:

\[ |\text{GHZ}_\theta^{(N)}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes N + e^{iN\theta} |1\rangle \otimes N) \]

\[ N = 1 \quad \text{and} \quad N \gg 1 \]

Faster phase accumulation. State periodicity \( T = \frac{2\pi}{N} \).
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For:

- \( N = 1 \)
- \( N \gg 1 \)

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N &= 1 \\
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Is the Heisenberg scaling reachable? Probes of different sizes:

\( |\psi_1\rangle \quad |\psi_1\rangle \quad |\psi_2\rangle \quad |\psi_2\rangle \quad |\psi_3\rangle \) \{ \text{total of } N \text{ quantum systems} \}
Kitaev Phase Estimation Algorithm

\[ |0\rangle \quad \cdots \quad |0\rangle + e^{2\pi i (2^{l-1} \varphi)} |1\rangle \]

\[ H^\otimes t \quad \cdots \quad |0\rangle + e^{2\pi i (2^1 \varphi)} |1\rangle \]

Bayesian estimation (Granade et al., New J. Phys. 14 103013 (2012))

The outliers make the estimator for the MSE unstable.

In both cases we cannot claim easily that the Heisenberg scaling can be reached.

**Solution:** analytical evaluation of the tail of the distribution!
Analytical control of the tails of $\hat{\theta}$

+ Exponentially growing size for the states $2^k$

The HS can be reached!

$\Delta^2 \hat{\theta} \sim \frac{C}{N^2}$


Our contributions:

- Optimization of the number of probes for each size and the size of the probes.
- Resource distribution in presence of noise (photon loss, non-unitary visibilities).
- Resource distribution for a small quantum enhancement.
- Exponentially growing sizes are not necessary.


Experiment done in collaboration with the group of prof. Fabio Sciarrino at La Sapienza in Rome, with an optical setup.
- No large size photonic entangled states (NOON states) are available.
- Multipass setups are not stable enough to simulate a high entangled state.
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**Phase encoded in high total angular momentum states of light!**

**Q-plates**: electrically activated liquid crystal (LC) half-wave plate with a topologically non trivial vortex configuration of the LC molecules.
A q-plate is locally an halfwave plate (HWP) but with axis different at each point.

\[ \alpha(\phi) = q\phi + \alpha_0 \]

Q-plates are able to manipulate the OAM through polarization.

\[
\begin{align*}
|L\rangle & \rightarrow e^{+2i\alpha(\phi)} |R\rangle = |R\rangle - m \\
|R\rangle & \rightarrow e^{-2i\alpha(\phi)} |L\rangle = |L\rangle + m
\end{align*}
\]

\[ m = 2q \]
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\[ |H\rangle \rightarrow \frac{|R\rangle |-m\rangle + |L\rangle |m\rangle}{\sqrt{2}} \rightarrow \frac{|R\rangle |+m\rangle + |L\rangle |m\rangle}{\sqrt{2}} \rightarrow \frac{|R\rangle |+m\rangle + e^{-2i(m+1)\theta} |L\rangle |-m\rangle}{\sqrt{2}} \]
\[ |R\rangle |+m\rangle + e^{-2i(m+1)\theta} |L\rangle |-m\rangle \quad \rightarrow \quad \frac{\sqrt{2}}{} \quad \rightarrow \quad |L\rangle |+m\rangle + e^{-2i(m+1)\theta} |R\rangle |-m\rangle \quad \rightarrow \quad \frac{\sqrt{2}}{} \]

\[ |H\rangle \rightarrow |R\rangle |-m\rangle + |L\rangle |+m\rangle \quad \rightarrow \quad \frac{\sqrt{2}}{} \quad \rightarrow \quad |R\rangle |+m\rangle + |L\rangle |-m\rangle \quad \rightarrow \quad \frac{\sqrt{2}}{} \quad \rightarrow \quad |R\rangle |+m\rangle + e^{-2i(m+1)\theta} |L\rangle |-m\rangle \quad \rightarrow \quad \frac{\sqrt{2}}{} \]
Details of the encoding and decoding stages:
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Total resource number:

\[ 2q_i + 1 = m_i + 1 = 1, 2, 11, 51.\]

\[ N = 2 \sum_{i=1}^{4} n_i (m_i + 1).\]

Number of photons for each q-plate (optimized strategy).

Topological charge of the q-plate, (quantum resource).
Mean Square Error results for many angles:
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Angular coefficients of the fit.
Convergence of $\hat{\theta}$ to $\theta$:

Asymptotic normality of $\hat{\theta}$
Outliers for the $\hat{\theta}$ estimator:

Convergence of $\hat{\theta}$ to $\theta$:

Asymptotic normality of $\hat{\theta}$
Thank you for your attention!