NANO COLLOQUIA 2022

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NEST, Scuola Normale Superiore

17 February 2022

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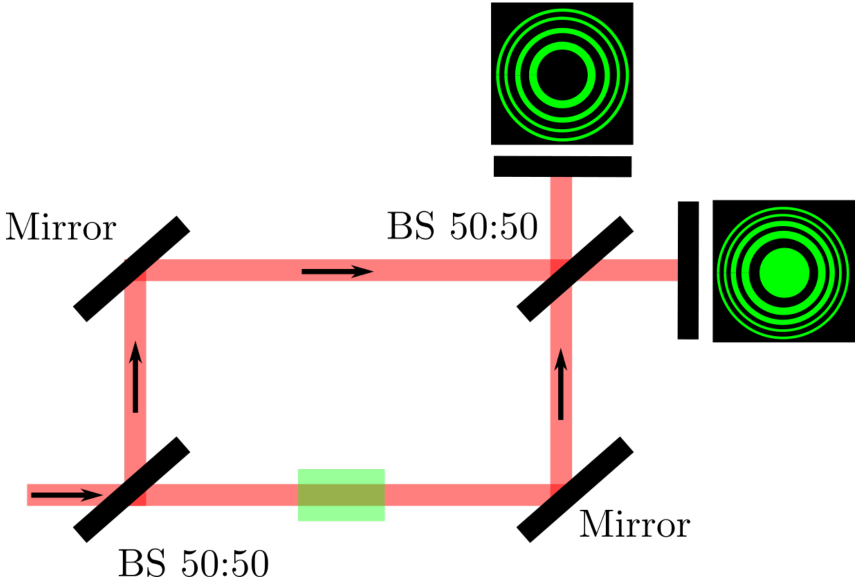
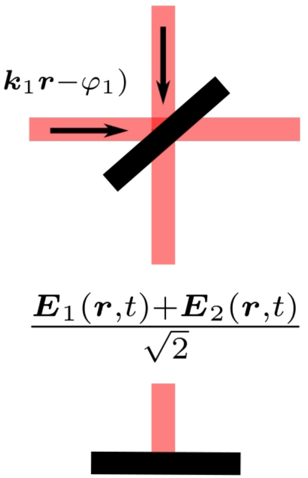
Non-asymptotic Heisenberg scaling: experimental metrology for a wide resources range

V Cimini, E Polino, F Belliardo, F Hoch, B Piccirillo, N Spagnolo, V Giovannetti, and F Sciarrino, arXiv:2110.02908 (2021).

The parameter to be estimated will be a phase (rotation angle).

Mach-Zehnder Interferometer: used to measure changes in the optical lengths of the arms.





$$\boldsymbol{E}_2(\boldsymbol{r},t) = \hat{\boldsymbol{E}}_2 e^{i(\omega t - \boldsymbol{k}_2\boldsymbol{r} - \varphi_2)}$$

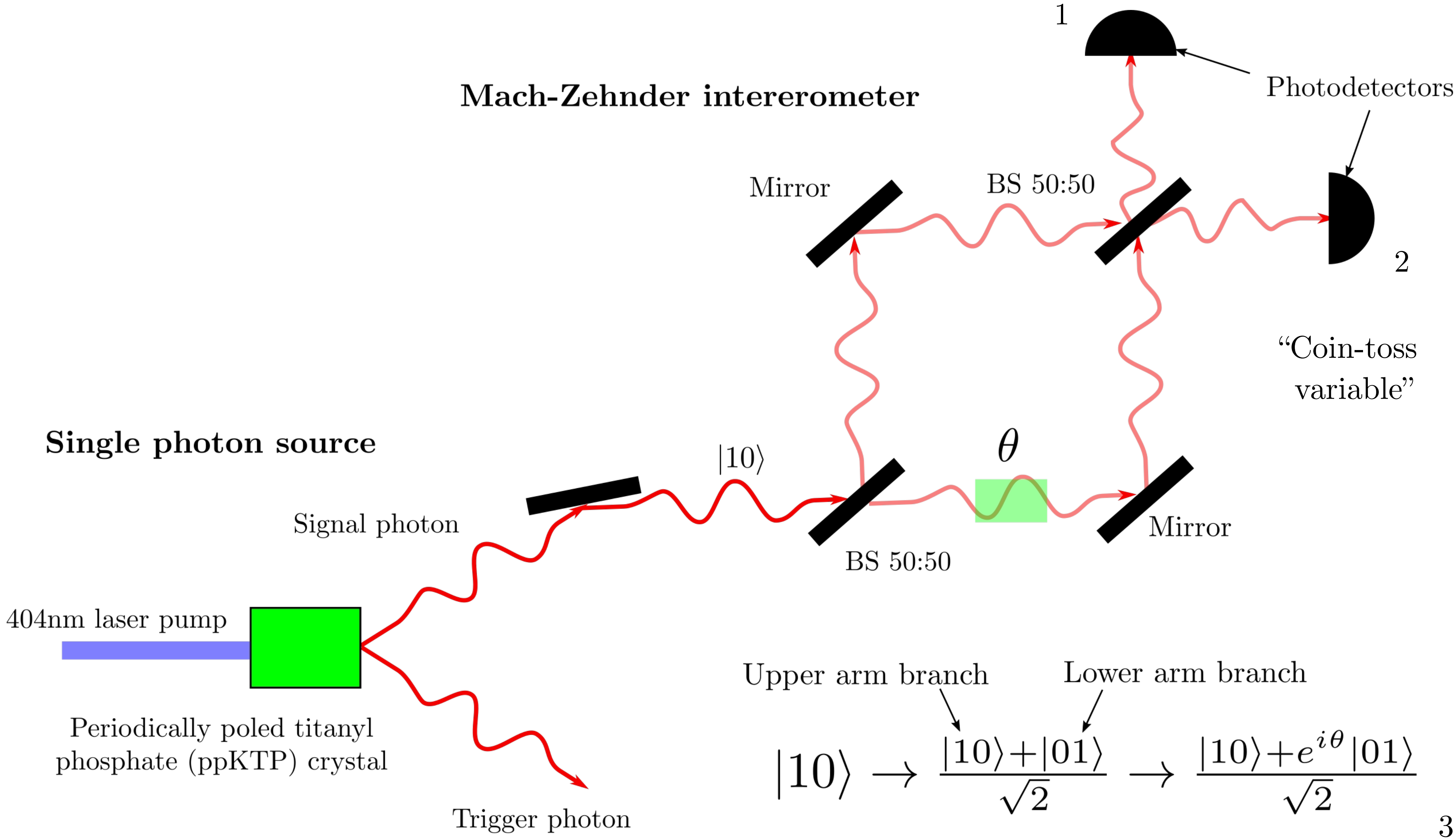
$$\boldsymbol{E}_1(\boldsymbol{r},t) = \hat{\boldsymbol{E}}_1 e^{i(\omega t - \boldsymbol{k}_1\boldsymbol{r} - \varphi_1)}$$

$$\frac{\boldsymbol{E}_1(\boldsymbol{r},t) - \boldsymbol{E}_2(\boldsymbol{r},t)}{\sqrt{2}}$$

$$\frac{\boldsymbol{E}_1(\boldsymbol{r},t) + \boldsymbol{E}_2(\boldsymbol{r},t)}{\sqrt{2}}$$

$$I = |\hat{\boldsymbol{E}}_1|^2 + |\hat{\boldsymbol{E}}_2|^2 \;\; + 2\hat{\boldsymbol{E}}_1\hat{\boldsymbol{E}}_2 \cos((\boldsymbol{k}_1 - \boldsymbol{k}_2) + \underbrace{\varphi_1 - \varphi_2}_{\text{Optical phase difference between the two arms.}})$$

What if we use truly quantum states of light? E.g. number states.


Mach-Zehnder intererometer
1
Photodetectors
Mirror
BS 50:50
2
"Coin-toss variable"
Single photon source
$|10\rangle$
θ
Signal photon
Mirror
BS 50:50
404nm laser pump
Periodically poled titanyl phosphate (ppKTP) crystal
Trigger photon
Upper arm branch
Lower arm branch
$$|10\rangle \rightarrow \frac{|10\rangle + |01\rangle}{\sqrt{2}} \rightarrow \frac{|10\rangle + e^{i\theta}|01\rangle}{\sqrt{2}}$$

Abstractly $\frac{|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}$ with $|0\rangle$ and $|1\rangle$ being two orthogonal quantum states.

Phase estimation is the prototypical quantum technological task!

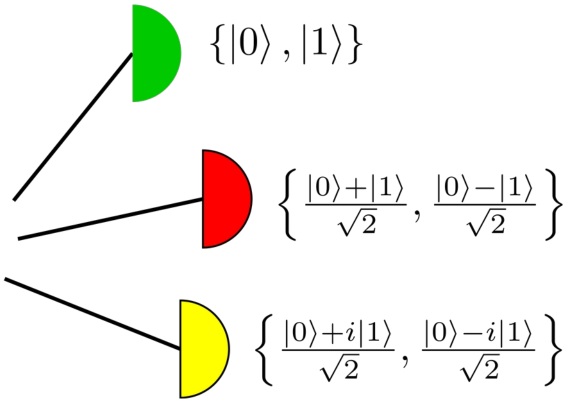
- Has its origin in the coherent nature of quantum mechanics, which allows for superposition with complex amplitudes.
- It's prototypical for the nature of quantum information:

  $\theta \in \mathbb{R}$ contains ∞ bits of information, but a measurement can extract at most 1 bit.

  We have however many ways to extract such bit, corresponding to many measurements.

Different points of view

$$|\psi_\theta\rangle = \frac{|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}$$



Roles of the Quantum Phase Estimation in...

Quantum Algorithms:

- Quantum Eigensolver
- HHL algorithm for linear systems
  (Harrow, Hassidim, and Lloyd, (2008)
- *Phys. Rev. Lett.* **103** (15): 150502)
- Shor factorization algorithm
  (Shor (1997), *SIAM J. Comput.*, **26** (5): 1484–1509)
- ...

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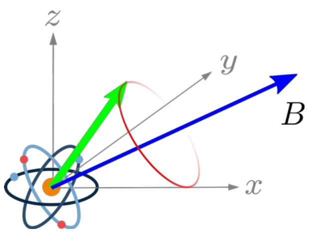
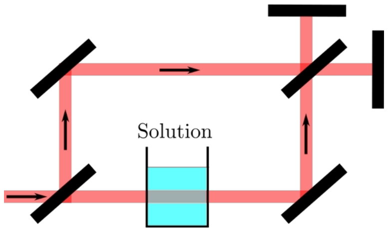
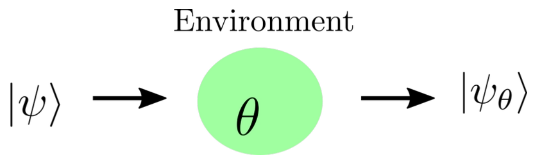
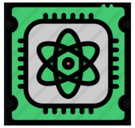




Quantum Metrology:



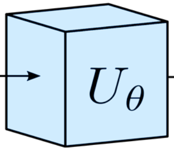
- Interferometry (gravitational waves)
  (Schnabel *et al.* *Nat Commun* **1**, 121 (2010))
- Concentration measurements in a solution
  (Taylor and Bowen, *Phys. Rep.* **615**, 1 (2016))
- Magnetometry
  (Budker and Romalis, Nature Physics **3**, 227–234 (2007))
- Characterization of quantum gates
- ...



Abstract point of view: single qubit encoded by a gate σ_z

$$U_\theta := e^{i\sigma_z\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \longrightarrow$$



$$\longrightarrow |\psi_\theta\rangle := U_\theta|\psi\rangle = \frac{|0\rangle+e^{i\theta}|1\rangle}{\sqrt{2}}$$

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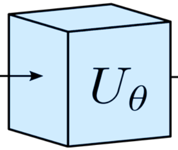
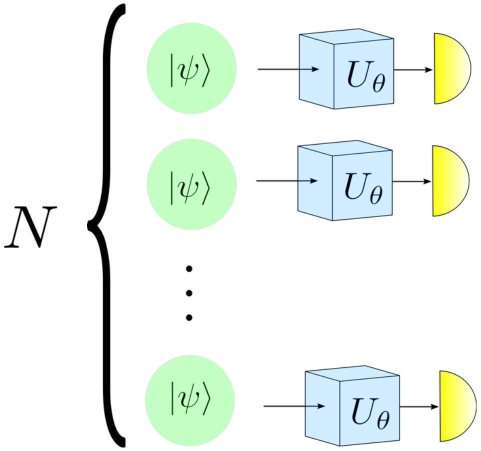


$$\longrightarrow |\psi_\theta\rangle := U_\theta|\psi\rangle = \frac{|0\rangle+e^{i\theta}|1\rangle}{\sqrt{2}}$$

We can define an estimator $\hat{\theta}$ from the results of multiple measurements, and define a **Quantum Cramér-Rao bound**.

1) Non-entangled probes:



$$\Delta^2\hat{\theta} = \mathbb{E}[(\hat{\theta}-\theta)^2] \geq \frac{1}{N}$$

Giovannetti, Lloyd, and Maccone, Phys. Rev. Lett. **96**, 010401 (2006).

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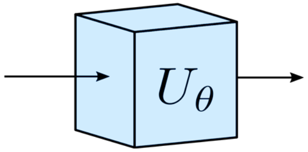

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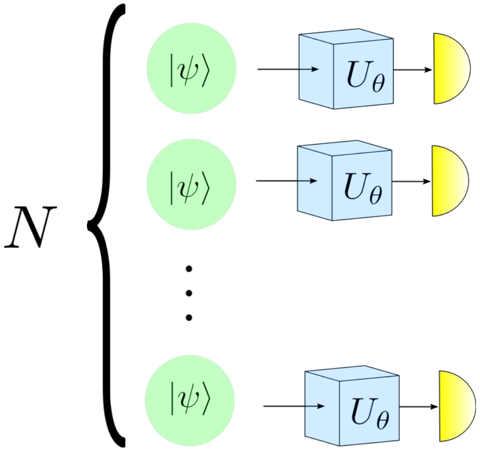

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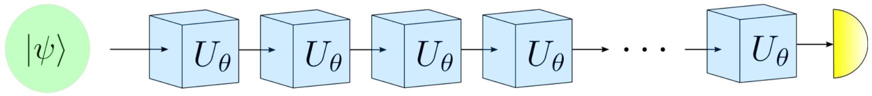
2) Sequential encoding:



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Heisenberg scaling:

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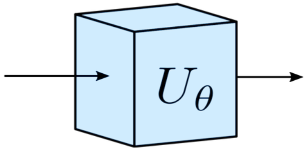

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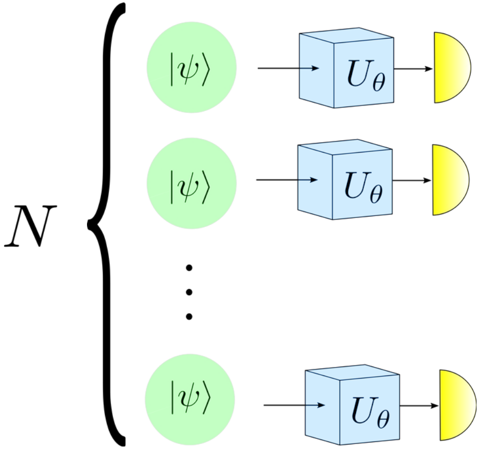

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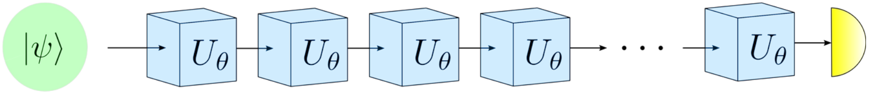

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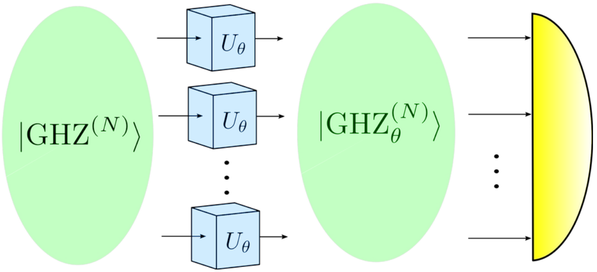
3) Entangled probes:



Heisenberg scaling:

$$\Delta^2\hat{\theta} \geq \frac{1}{N^2}$$

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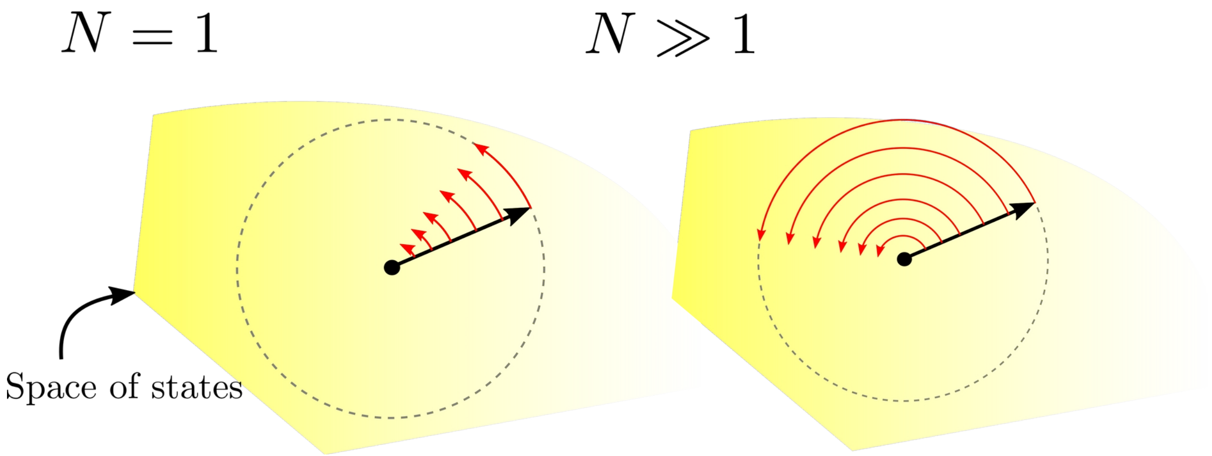
Maximally entangled states for Quantum Metrology: $|\mathrm{GHZ}_\theta^{(N)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + e^{iN\theta}|1\rangle^{\otimes N})$



Faster phase accumulation. State periodicity $T = \frac{2\pi}{N}$.

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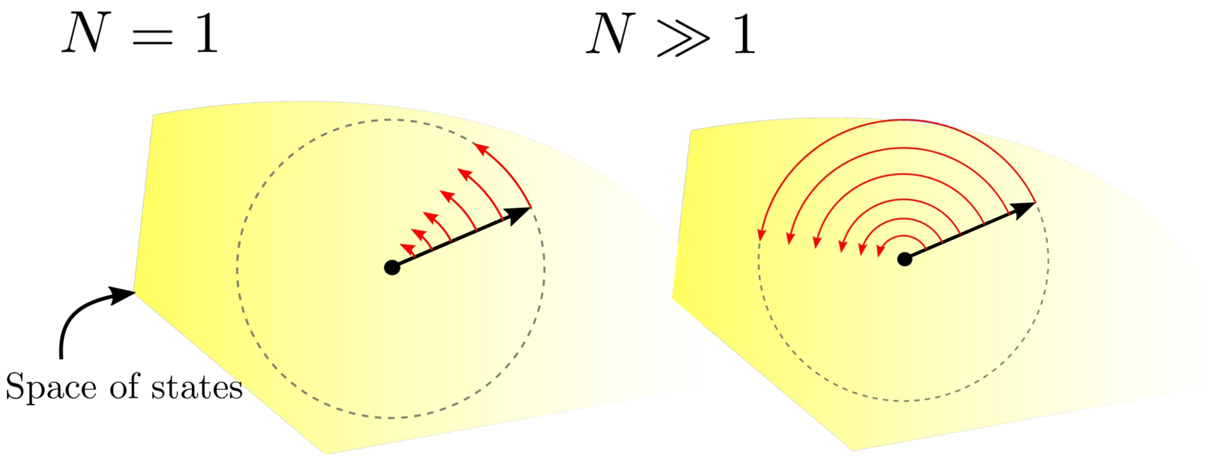

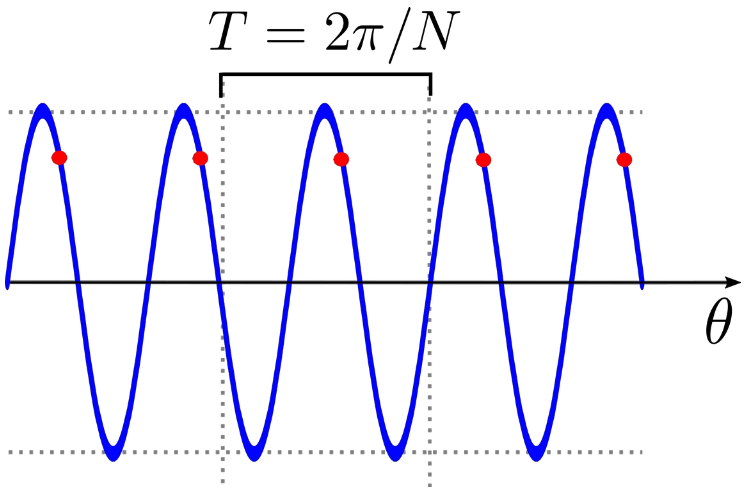

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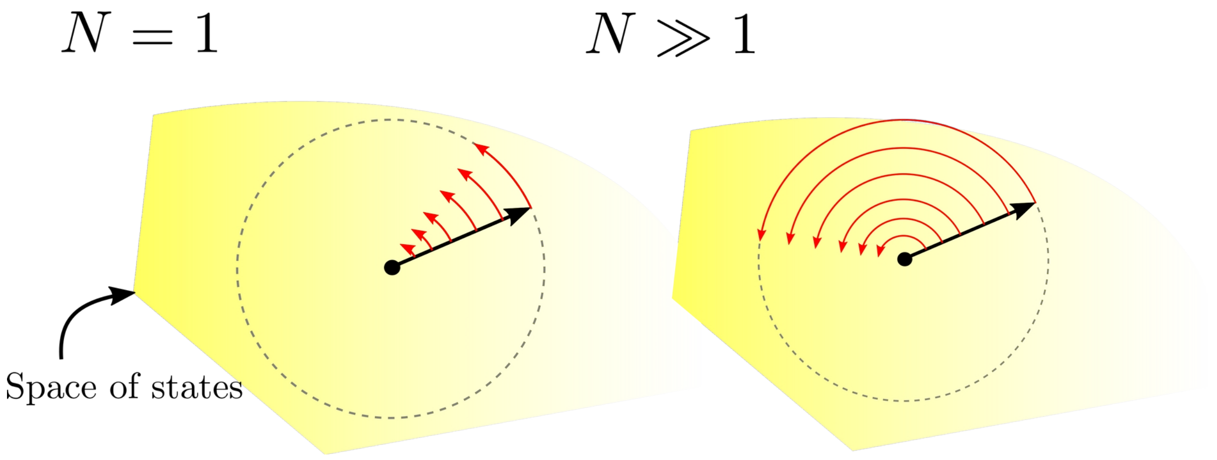

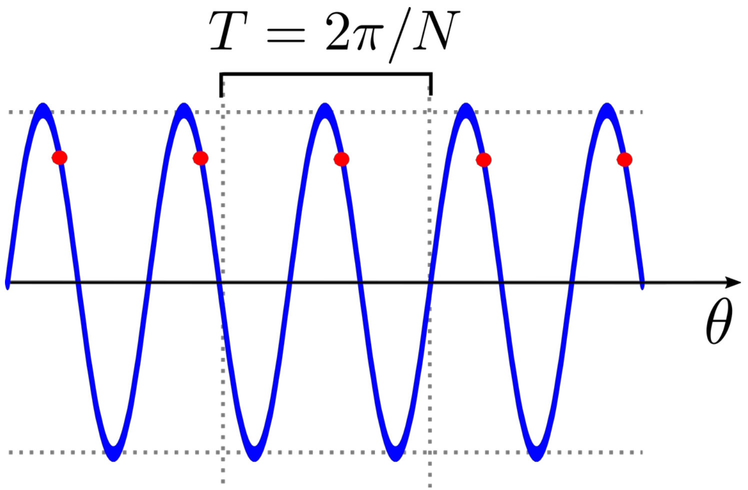

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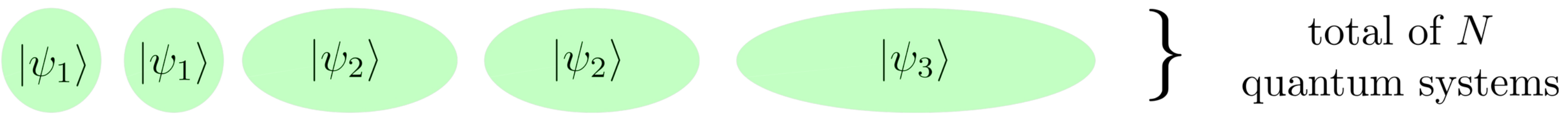
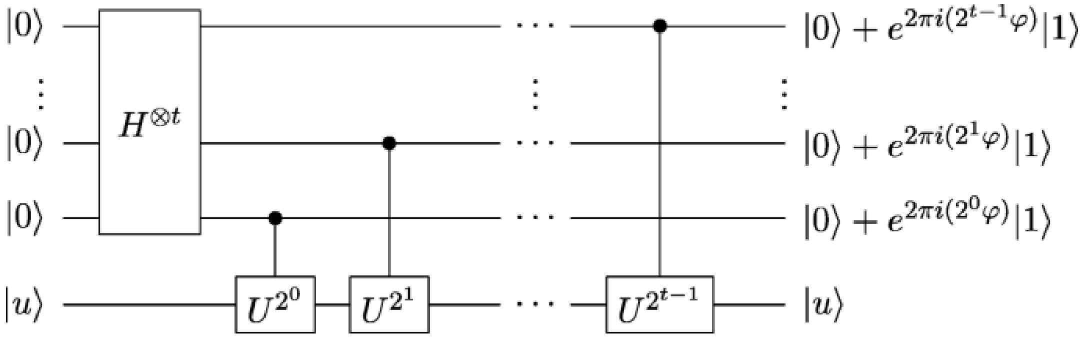
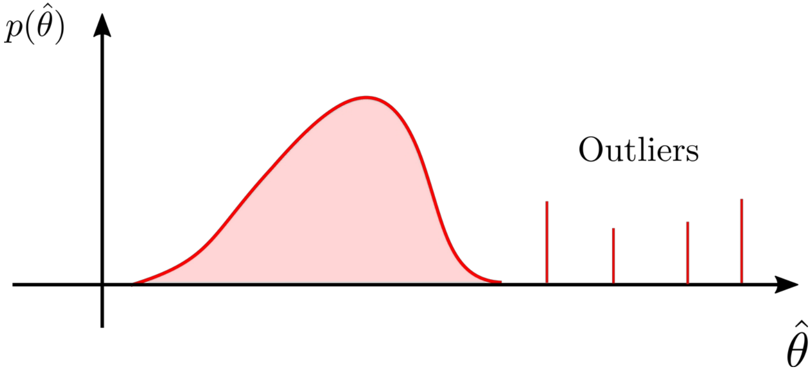
Is the Heisenberg scaling reachable? Probes of different sizes:



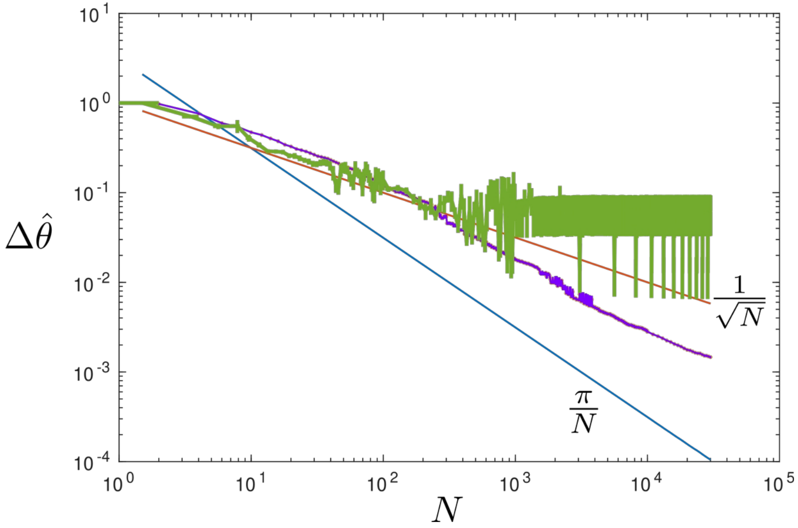
Kitaev Phase Estimation Algorithm





The outliers ruin the MSE $\rightarrow \Delta^2\hat{\theta} \sim \frac{1}{N}$

Bayesian estimation (Granade *et al.*, *New J. Phys.* 14 103013 (2012))



The outliers make the estimator for the MSE unstable.

In both cases we cannot claim easily that the Heisenberg scaling can be reached.

Solution: analytical evaluation of the tail of the distribution!

Analytical control of the tails of $\hat{\theta}$
$+$
Exponentially growing size for the states 2^k

→ **The HS can be reached!**

$$\Delta^2\hat{\theta} \simeq \frac{C}{N^2}$$

Higgins, Berry, Bartlett, Mitchell, Wiseman, and Pryde, *New J. Phys.* **11**, 073023 (2009).
Kimmel, Low, and Yoder, *Phys. Rev. A* **92**, 062315 (2015).

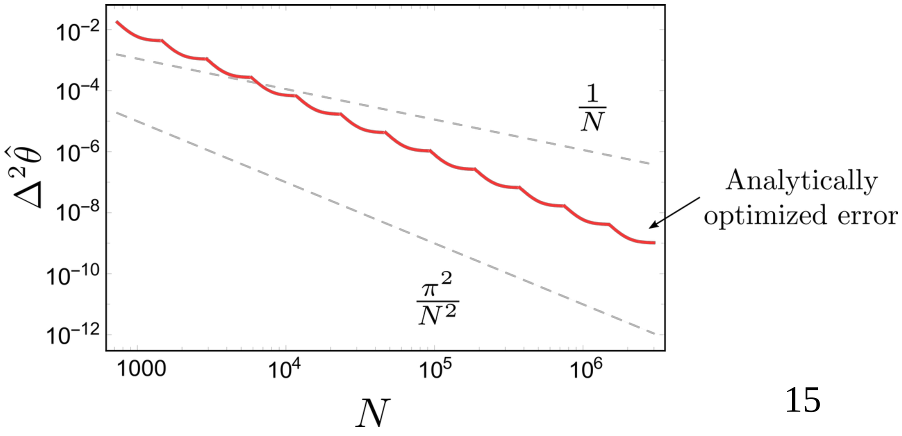
Our contributions:

- **Optimization of the number of probes for each size and the size of the probes.**
- Resource distribution in presence of noise (photon loss, non-unitary visibilities).
- Resource distribution for a small quantum enhancement.
- Exponentially growing sizes are not necessary.

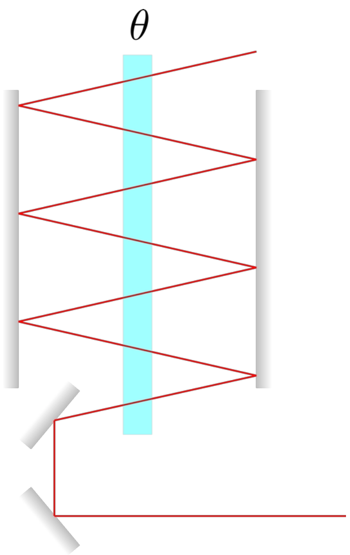
F. Belliardo and V. Giovannetti, *Phys. Rev. A* **102**, 042613 (2020)

↓

Experiment done in collaboration with the group of prof. Fabio Sciarrino at La Sapienza in Rome, with an optical setup.



- No large size photonic entangled states (NOON states) are available.
- Multipass setups are not stable enough to simulate a high entangled state.



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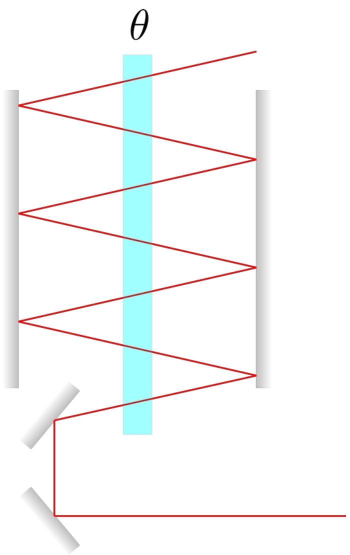
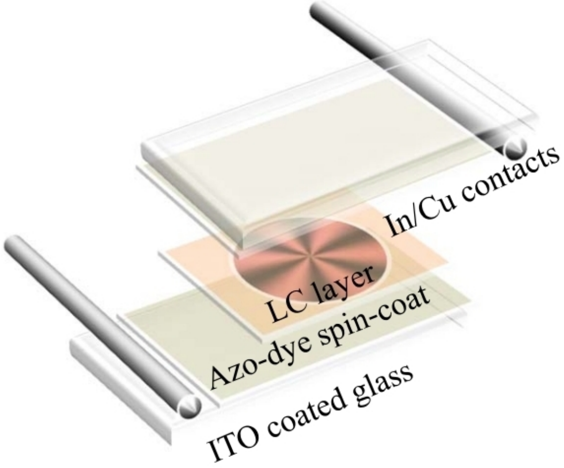
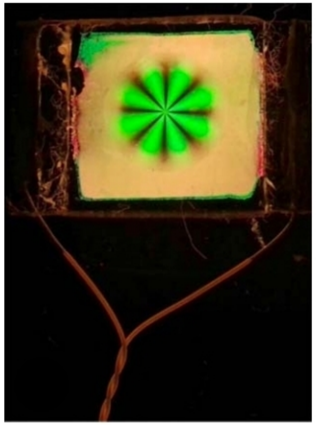
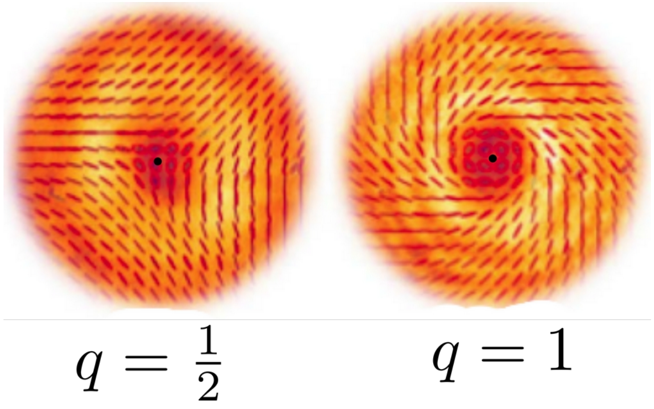


Phase encoded in high total angular momentum states of light!

Q-plates: electrically activated liquid crystal (LC) half-wave plate with a topologically non trivial vortex configuration of the LC molecules.







Rubano, Cardano, Piccirillo, and Marucci, *Journal of the Optical Society of America B* **36**, Issue 5, pp. D70-D87 (2019).

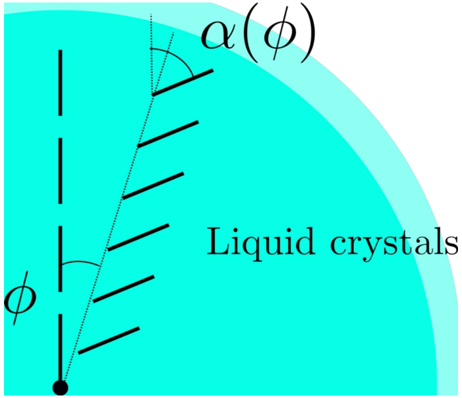
A q-plate is locally an halfwave plate (HWP) but with axis different at each point.

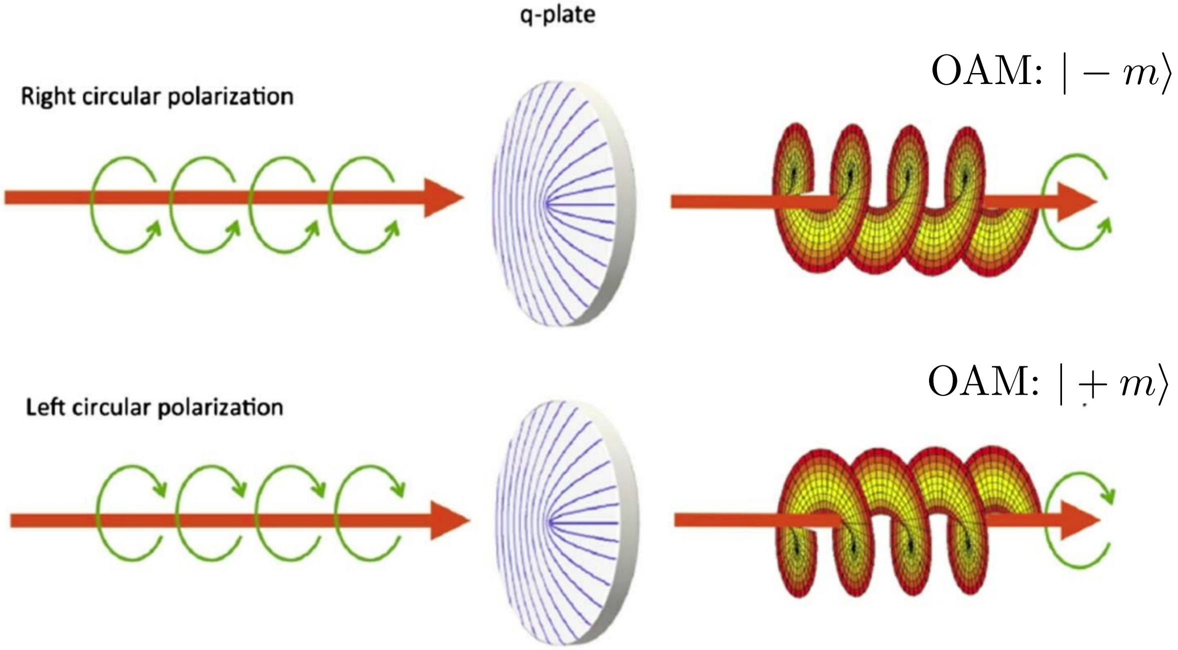
$$\alpha(\phi) = q\phi + \alpha_0$$

Q-plates are able to manipulate the OAM through polarization.

$$\begin{cases} |L\rangle \to e^{+2i\alpha(\phi)}|R\rangle = |R\rangle|-m\rangle \\ |R\rangle \to e^{-2i\alpha(\phi)}|L\rangle = |L\rangle|+m\rangle \end{cases}$$

$$m = 2q$$





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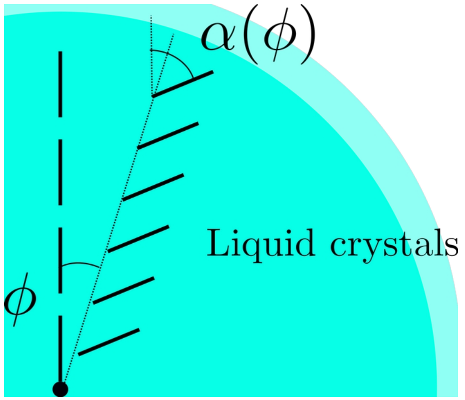

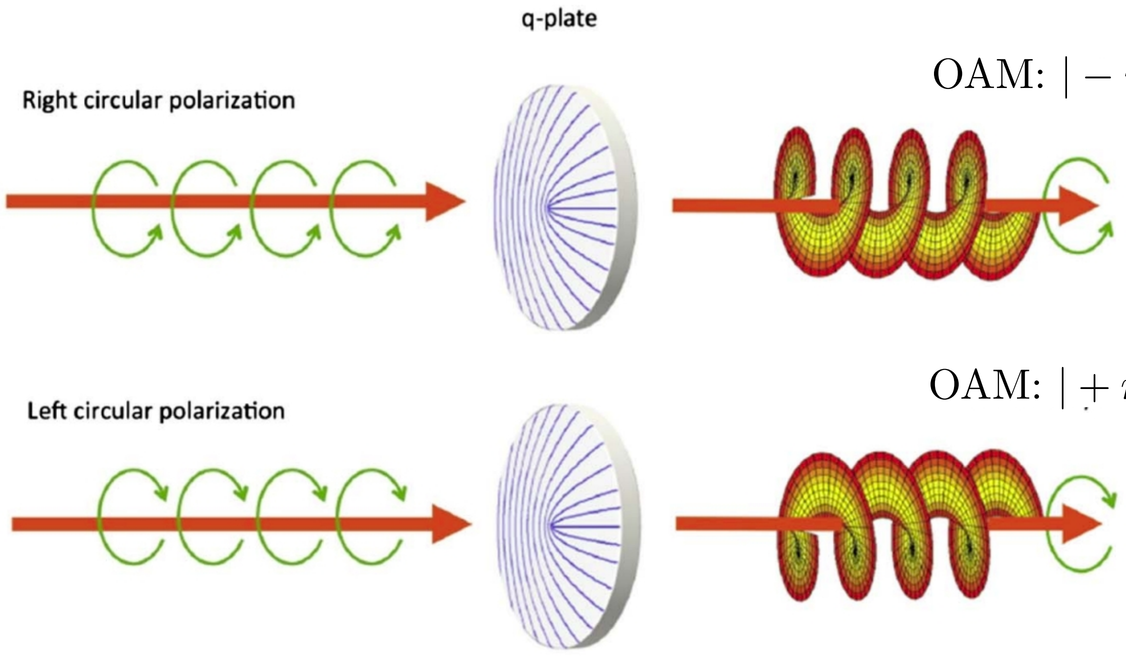

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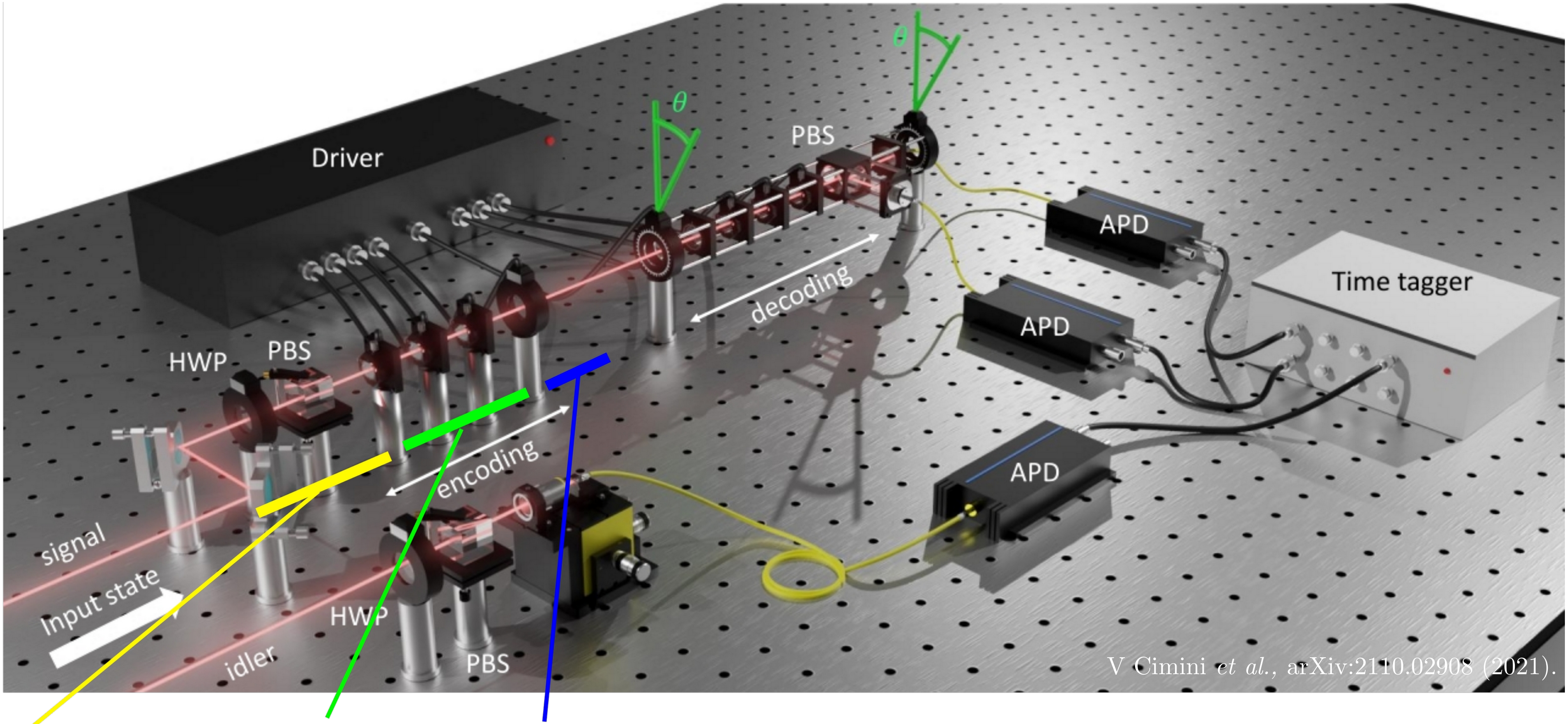
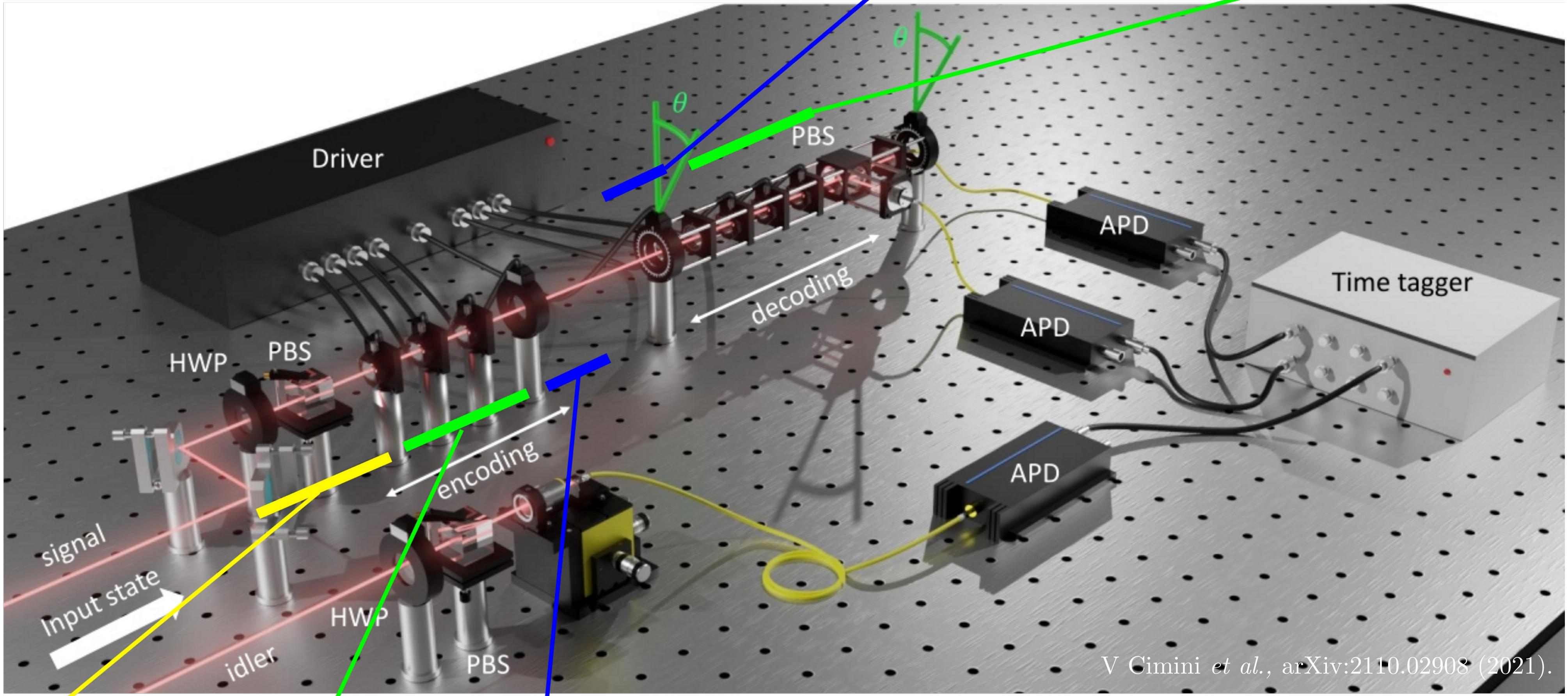


$$|H\rangle \rightarrow \frac{|R\rangle\,|-m\rangle + |L\rangle\,|+m\rangle}{\sqrt{2}} \rightarrow \frac{|R\rangle\,|+m\rangle + |L\rangle\,|-m\rangle}{\sqrt{2}} \rightarrow \frac{|R\rangle\,|+m\rangle + e^{-2i(m+1)\theta}\,|L\rangle\,|-m\rangle}{\sqrt{2}}$$

$$\frac{|R\rangle|+m\rangle + e^{-2i(m+1)\theta}|L\rangle|-m\rangle}{\sqrt{2}} \rightarrow \frac{|L\rangle|+m\rangle + e^{-2i(m+1)\theta}|R\rangle|-m\rangle}{\sqrt{2}} \rightarrow \frac{|L\rangle + e^{-2i(m+1)\theta}|R\rangle}{\sqrt{2}}$$



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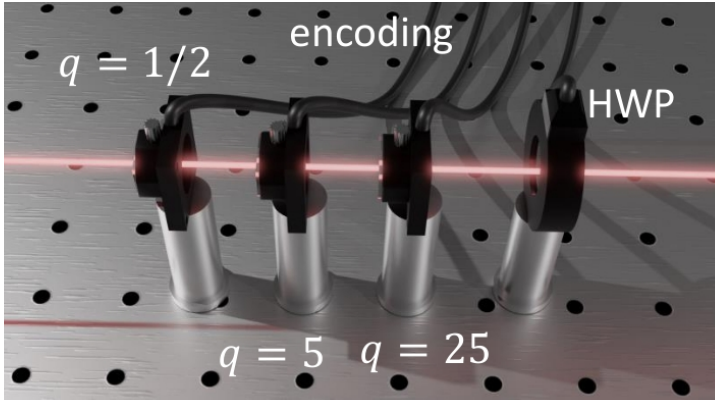
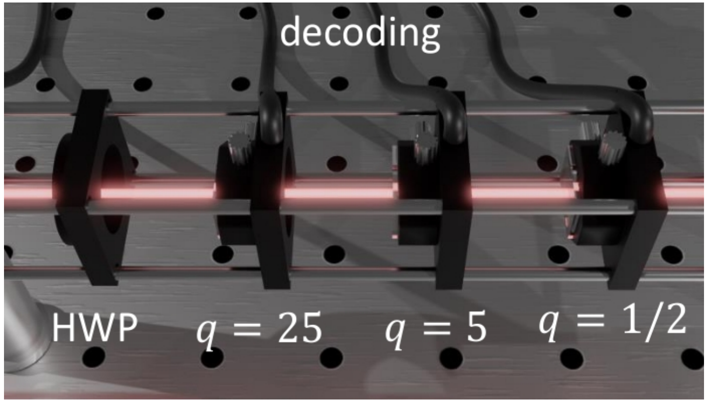
Details of the encoding and decoding stages:





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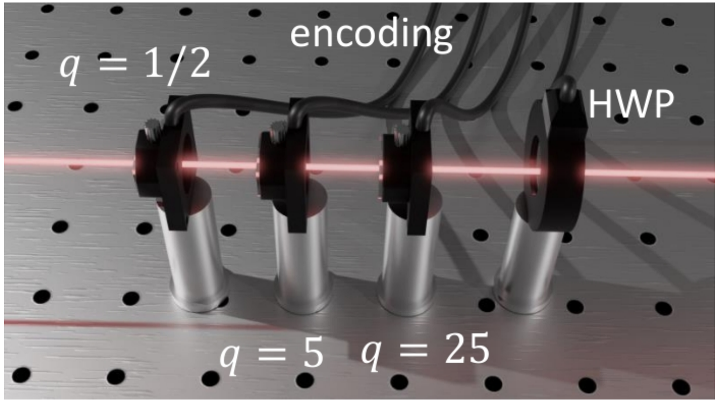
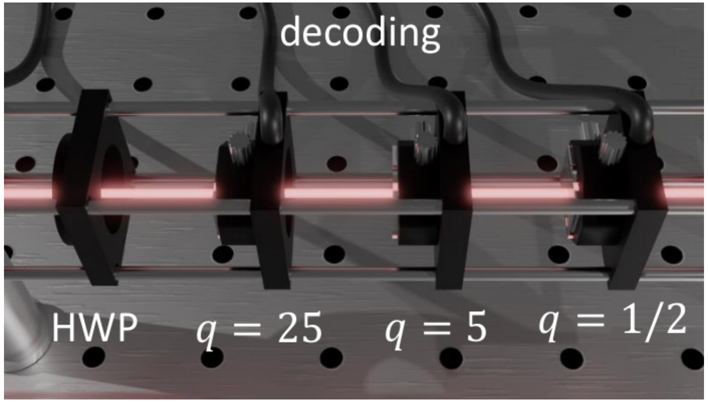




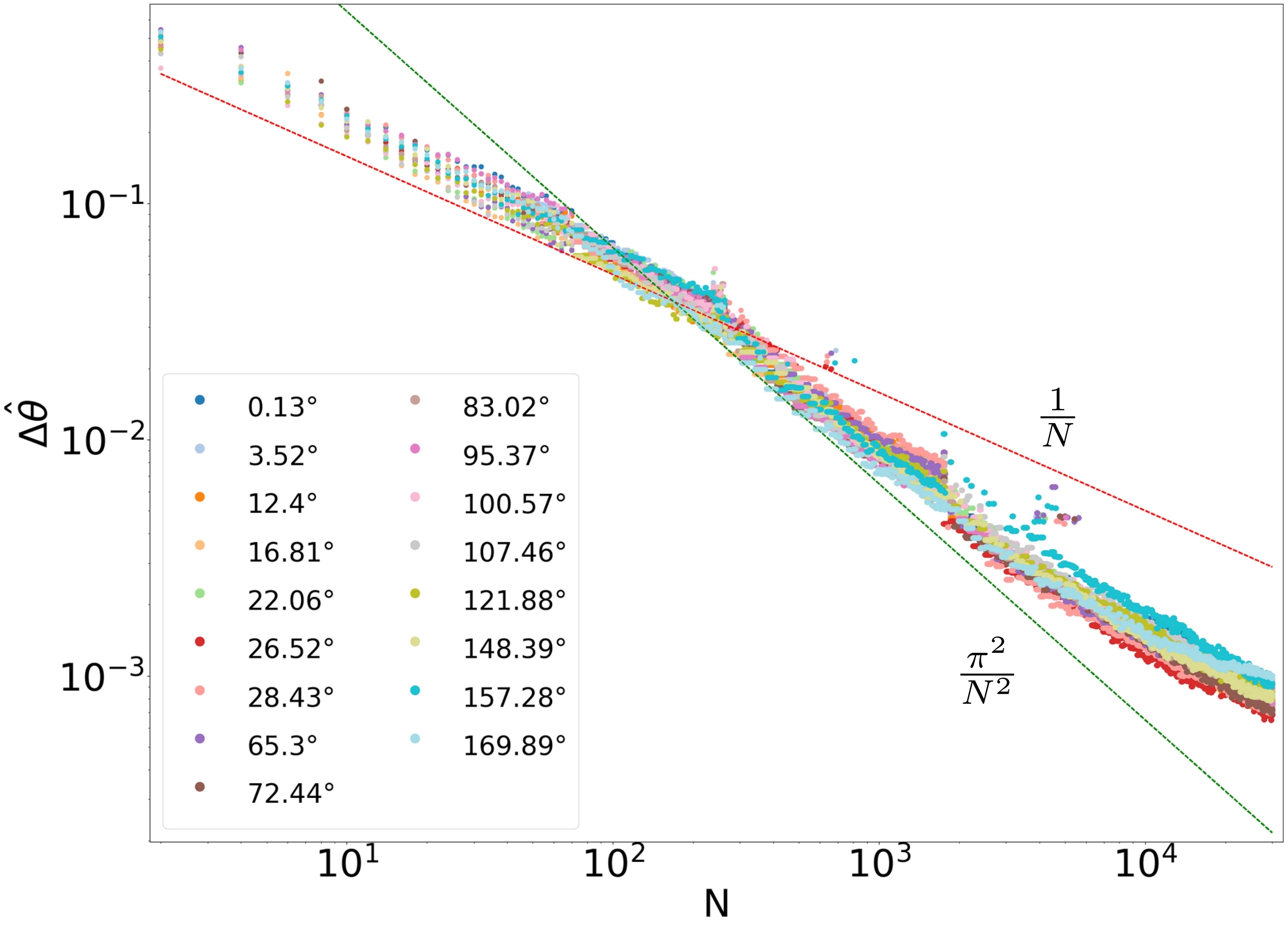
Total resource number:

Topological charge of the q-plate,
(quantum resource).

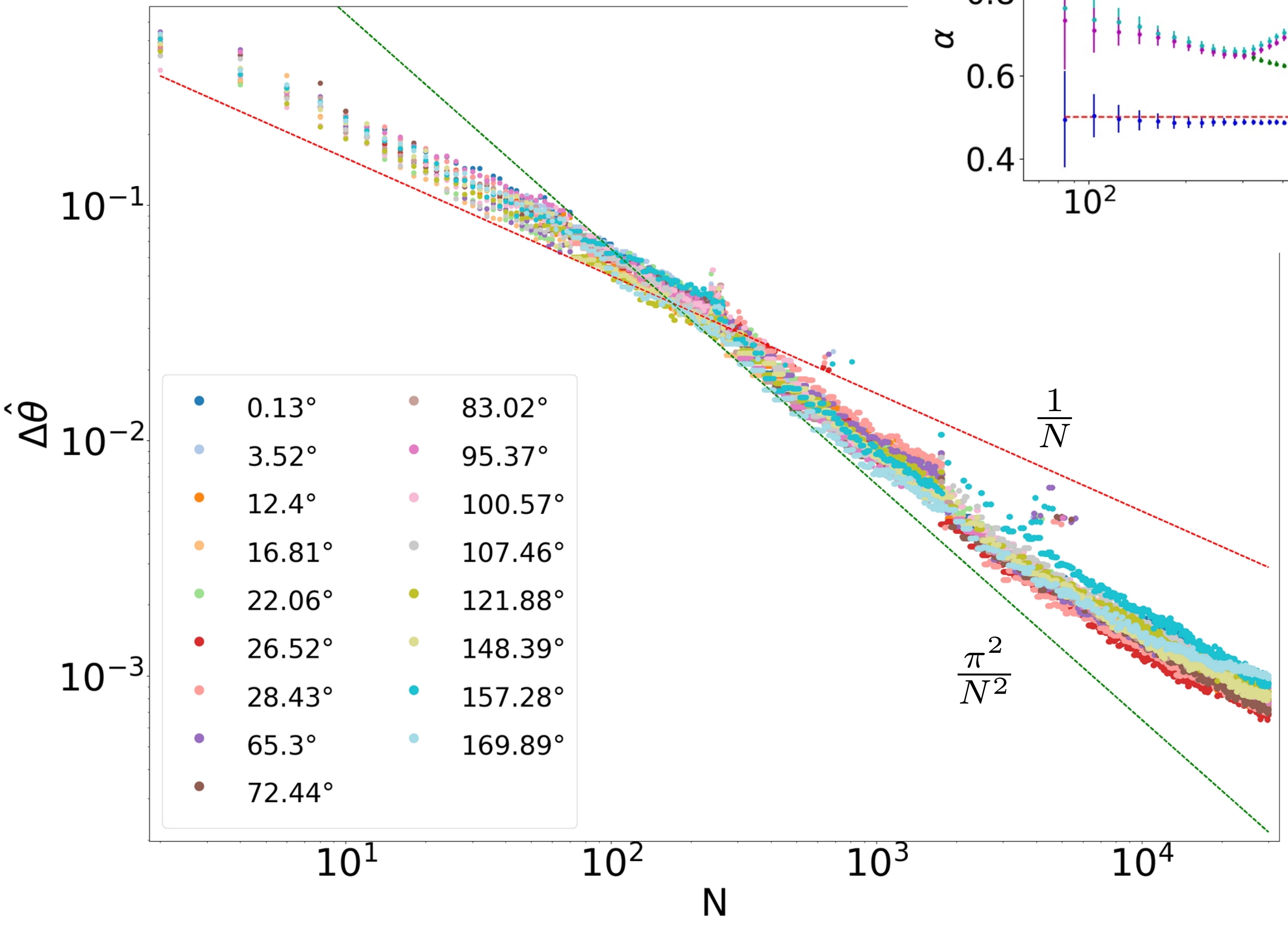
$$2q_i + 1 = m_i + 1 = 1, 2, 11, 51 \; . \qquad N = 2 \sum_{i=1}^{4} n_i (m_i + 1) \; .$$

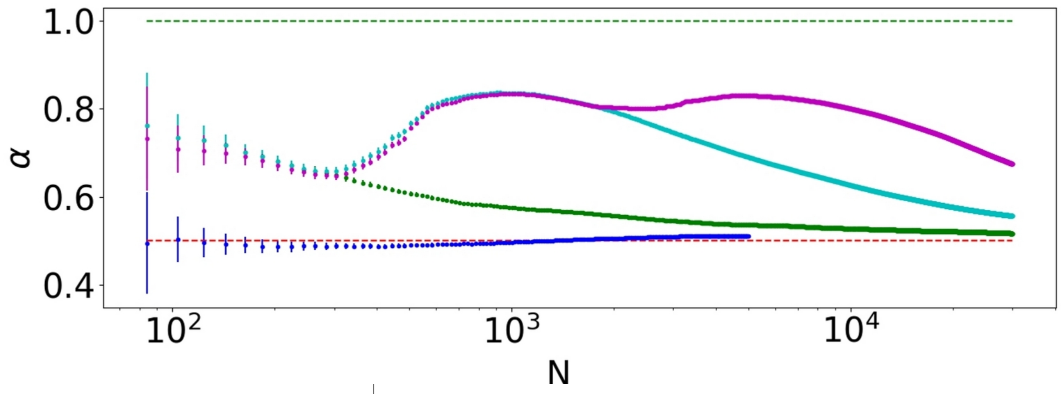
Number of photons
for each q-plate (optimized strategy).

Mean Square Error results for many angles:



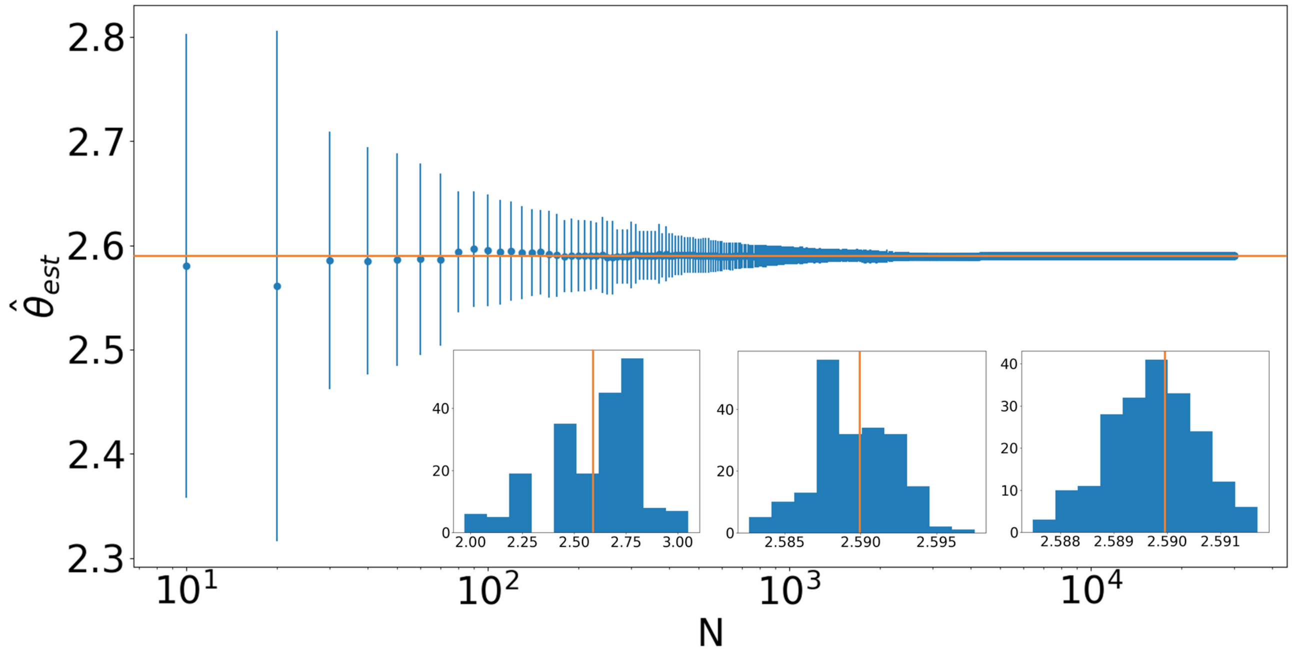
Mean Square Error results for many angles:





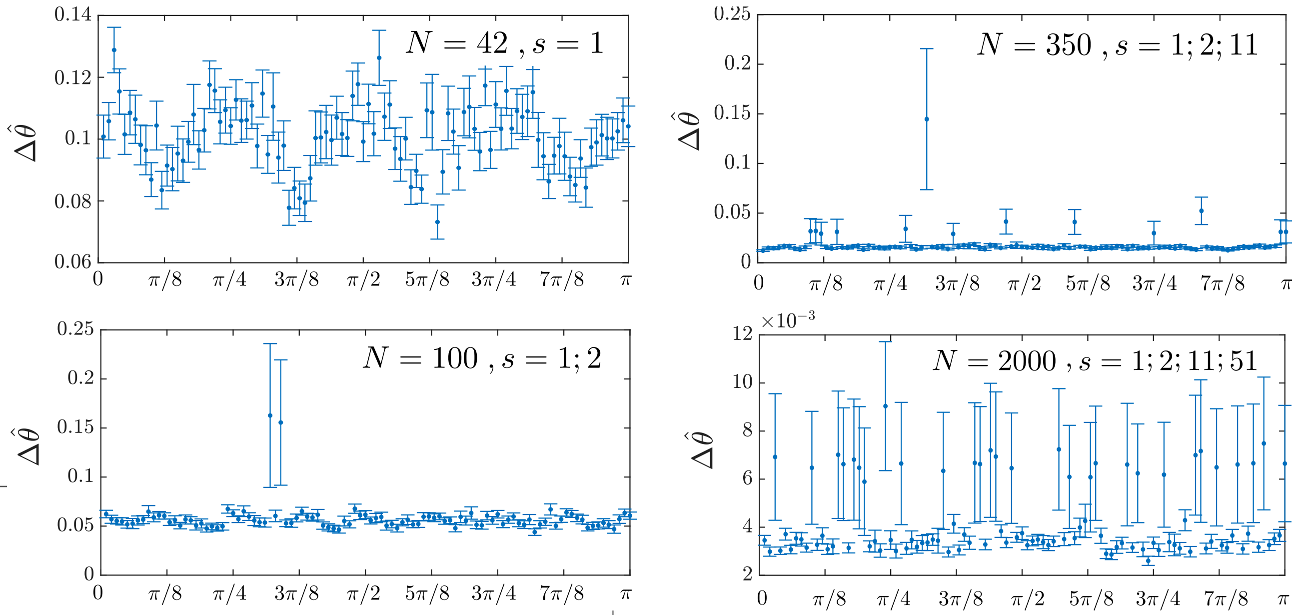
Angular coefficients of the fit.

Convergence of $\hat{\theta}$ to θ:

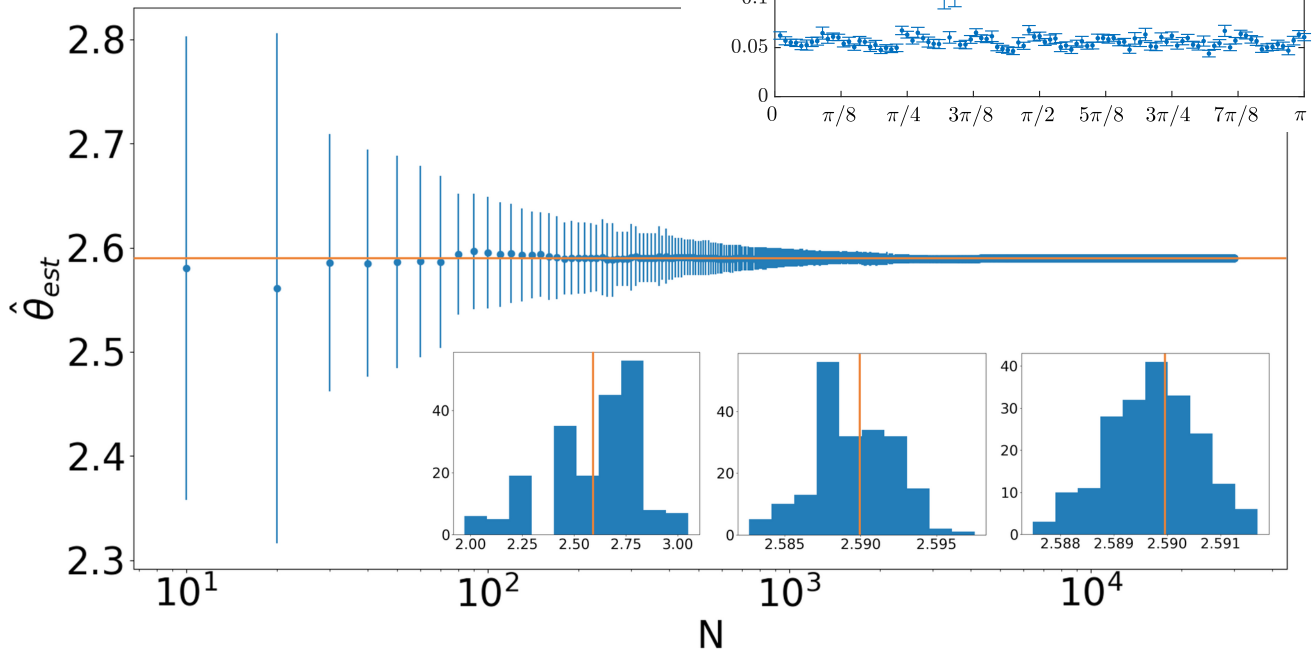


Asymptotic nomality of $\hat{\theta}$

Outliers for the $\hat{\theta}$ estimator:



Convergence of $\hat{\theta}$ to θ:



Asymptotic nomality of $\hat{\theta}$

Thank you for your attention!